Math 2010 B Tutorial 4 2020/10

Outline

- Exercise for taking limit

Excise : whether the following limit exist or not ?

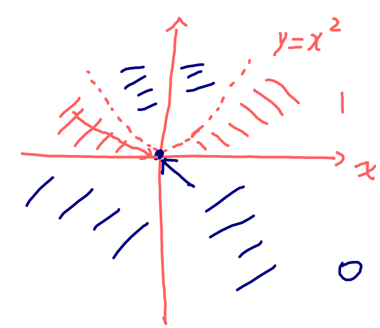
① $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} (x+1) = 2$ ← basic properties & continuous funct preserve limits

② $\lim_{(x,y)\to(0,0)} \frac{\overbrace{x^3+2xy^2}^{\text{degree } 3}}{\underbrace{x^2+y^2}_{\text{degree } 2}} = 0$ ← using polar coordinate

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $r \to 0$

③ $\lim_{x\to 0} \frac{\sin x}{x}$ ← L'Hopital Rule $= \lim_{x\to 0} \frac{\cos x}{1} = 1$

④ $f(x,y) = \begin{cases} 1 & , \text{if } 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$



$\lim_{(x,y)\to(0,0)} f(x,y)$ not exist

Rk: limit not exist $\begin{cases} \varepsilon-\delta \text{ language} \\ 2\text{-path test} \end{cases}$

Techniques for Evaluating Limit

· $\exists$ of limit & computing it.

1) By def (ε-δ argument, $\forall \varepsilon>0, \exists \delta \cdots\cdots$)

2) By basic properties of limit

$\pm$, $\times$, $(\cdot)^n$ etc

(In general, continuous function preserve limits.

(i.e if $f(x)$ is continuous at $x=x_1$

then $\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x) = f(x_0)$)

$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1$

3) By Squeeze Thm

4) By L'Hopital Rule

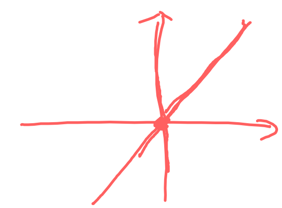
5) By taking $\lim_{r\to 0^+}$ in polar coord. (r,θ)

for calculating $\lim_{(x,y)\to(0,0)}$ on $\mathbb{R}^2$.

$x = r\cos\theta$

$y = r\sin\theta$.

· $\nexists$ of limit



1) By def ($\exists \varepsilon>0, \forall \delta>0, \cdots\cdots$)

2) By finding different paths to get unequal limits.

$y=x$

$y=1$

$x=0$

Define $f(x,y) = \begin{cases} 1, & \text{if } 0 < y < x^2 . \\ 0, & \text{otherwise} . \end{cases}$ ✩

$\lim_{(x,y)\to(0,0)} f(x,y)$

$\vec{a}$

Find $\lim_{(x,y)\to\vec{a}} f(x,y)$ where

i) $\vec{a} = (0,1)$ ii) $\vec{a} = (1,1)$;   iii) $\vec{a} = (0,0)$

$\boxed{Q : \lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist. (By } \varepsilon\text{-}\delta \text{ argument) ?}}$

$\boxed{\text{Recall}}$ ▷ $\lim_{(x,y)\to\vec{a}} f(x,y) \overset{=0}{}$ exist, by ε-δ argument denote limit by L.

$\forall\ \varepsilon > 0\ \exists\ \delta > 0$, s.t $\forall\ 0 < \|(x,y) - \vec{a}\| < \delta$, Then $|f(x,y) - L| < \varepsilon$. $\frac{1}{10}, \frac{1}{100}, \frac{1}{10000}$

$|f(x,y) - 0| < \varepsilon$

i) $\boxed{\vec{a} = (0,1)}$

for any $\varepsilon > 0$, take $\delta = \frac{1}{\sqrt{2}} > 0$ $\|(x, y-1)\|$

Consider any $(x,y) \in \mathbb{R}^2$ s.t $\boxed{0 < \|(x,y) - \vec{a}\| < \delta}$. ① ✩ $\Rightarrow$ $y \geq x^2$ $\Rightarrow$ $f(x,y) = 0$

want to show $y \geq x^2$ $(\Rightarrow f(x,y) = 0)$

We have $0 < (\sqrt{x^2 + (y-1)^2})^2 < \delta^2$ ② — $(*)$ ($\sqrt{x^2+(y-1)^2} = \|(x,y)-\vec{a}\|$)

Then $x^2 \leq x^2 + (y-1)^2 < \delta^2$

Rearranging $(*)$ $y > \frac{1}{2}(x^2 + y^2 + 1 - \delta^2)$ ✩   ($\delta^2 = \frac{1}{2}$)

$y^2 \geq 0 \Rightarrow \geq \frac{1}{2}(x^2 + 1 - \delta^2) = \frac{1}{2}(x^2 + \delta^2) \geq x^2$ ($1 - \delta^2 = \frac{1}{2}$)

$\Rightarrow y \geq x^2$

$|f(x,y) - 0| = |0 - 0| = 0 < \varepsilon$ (L)   ✩

$\Rightarrow$ $\therefore \lim_{(x,y)\to\vec{a}} f(x,y) = 0$

ii) $\vec{a} = (1,1)$ Assume $\lim_{(x,y)\to\vec{a}} f(x,y)$ exists & denote it by L

i.e $\forall\ \varepsilon = \frac{1}{2} > 0$

$\exists\ \delta > 0$ s.t $\forall\ (x,y)$ s.t "$0 < \|(x,y) - \vec{a}\| < \delta$",

$|f(x,y) - L| < \varepsilon$.

Remark: $\lim_{(x,y)\to\vec{a}} f(x,y) \neq L$

$\exists\ \varepsilon > 0,\ \forall \delta > 0$

$\exists!\,(x,y)$ s.t $0 < \|(x,y) - \vec{a}\| < \delta$

s.t $|f(x,y) - L| \geq \varepsilon$.

putting $(x,y) = (1 + \frac{\delta}{2}, 1)$ $\Leftrightarrow$ $x = 1 + \frac{\delta}{2},\ y = 1$

$\because$ $\begin{cases} 0 < \|(x,y) - \vec{a}\| < \frac{\delta}{2} < \delta \\ 0 < y = 1 < (1 + \frac{\delta}{2})^2 = x^2 \end{cases}$ $\Rightarrow y < x^2 \Rightarrow f(x,y) = 1$

$\therefore$ $|1 - L| = |f(x,y) - L| < \frac{1}{2}$ (1)

" for $(x,y) = (1 + \frac{\delta}{2}, 1)$

putting "$(x,y) = (1 - \frac{\delta}{2}, 1)$"

$\because$ $\begin{cases} 0 < \|(x,y) - \vec{a}\| = \frac{\delta}{2} < \delta \\ y = 1 > (1 - \frac{\delta}{2})^2 = x^2 \end{cases}$ $\Rightarrow y > x^2 \Rightarrow f(x,y) = 0$ (2)

$\therefore$ $|0 - L| = |f(x,y) - L| < \frac{1}{2}$

$\overset{\shortparallel}{0}$ for $(x,y) = (1 - \frac{\delta}{2}, 1)$

But (1) & (2) leads to contradiction $\Rightarrow$ $\lim_{(x,y)\to\vec{a}} f(x,y)$ does not exist